# Shadowing data assimilation for imperfect models

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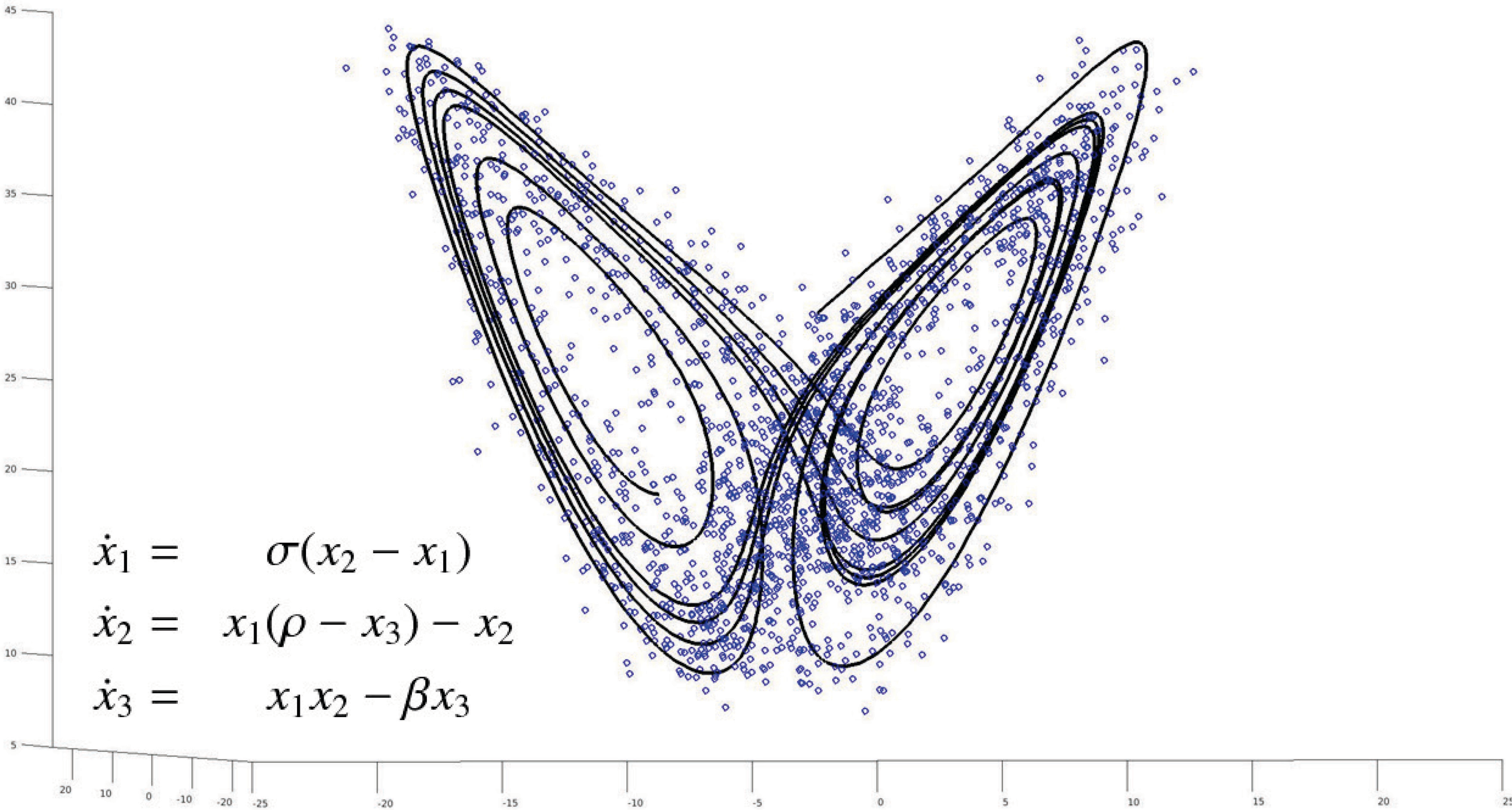
# Problem setting




$$
\begin{aligned}
\dot{x}_1 &= \sigma(x_2 - x_1) \\
\dot{x}_2 &= x_1(\rho - x_3) - x_2 \\
\dot{x}_3 &= x_1 x_2 - \beta x_3
\end{aligned}
$$

# Variatonal data assimilation

Given:

$$\text{Model} \quad x_{n+1} = F(x_n), \quad x_n \in \mathbb{R}^d,$$

$$\text{Observations} \quad y_n = H(X_n) + \eta_n, \quad y_n \in \mathbb{R}^{b \le d}, \quad n = -M, \ldots, N.$$

Find:

$$\mathbf{u} = \{u_{-M}, u_{-M+1}, \ldots, u_N\}, u_n \in \mathbb{R}^d, n = -M, \ldots, N$$

$$\text{such that} \quad ||y_n - H(u_n)|| \quad \text{small}$$

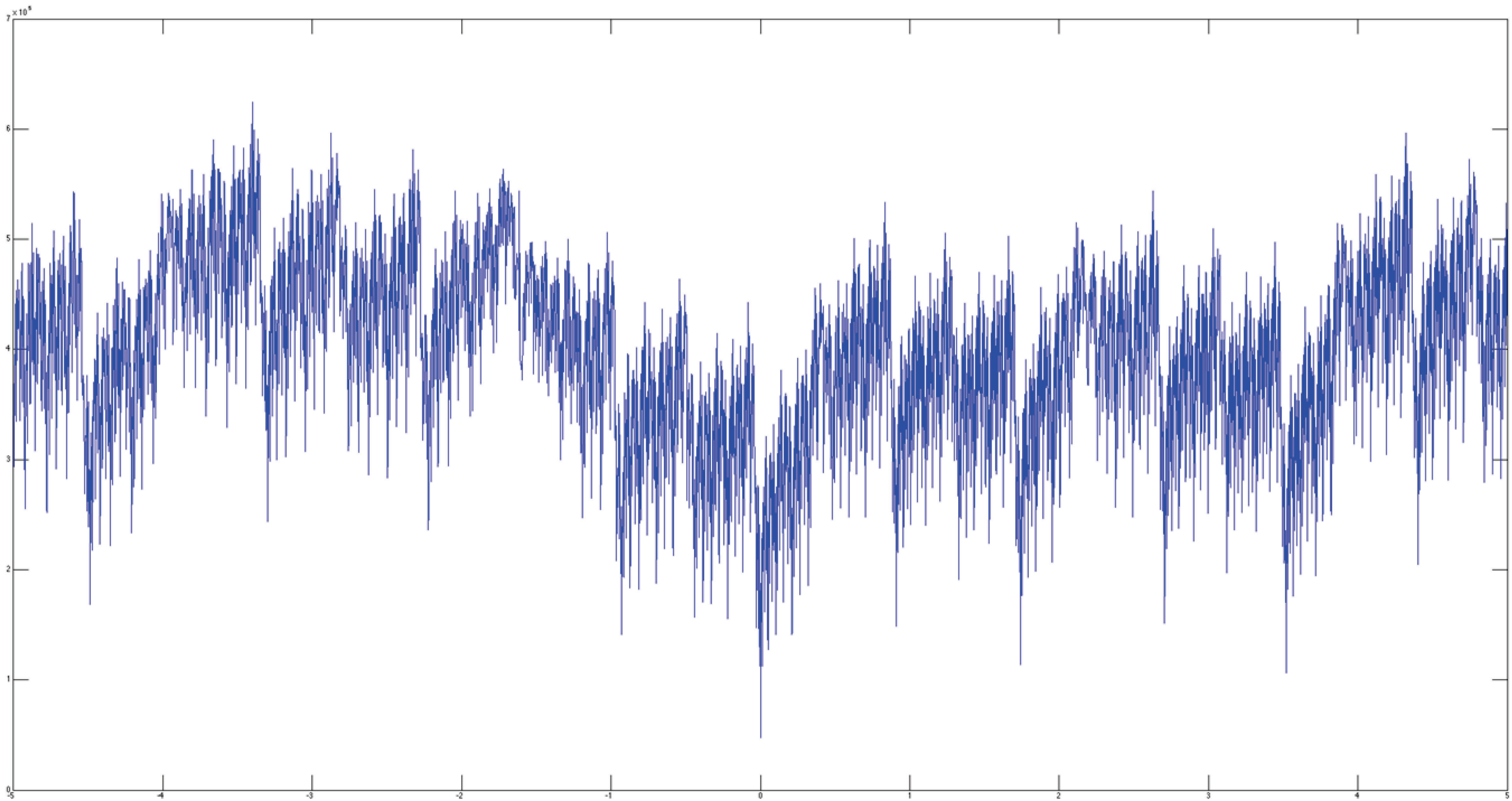
$$||G(\mathbf{u})|| = ||(G_{-M} G_{-M+1} \ldots G_{N-1})^T|| = 0$$

$$\text{with} \quad G_n(\mathbf{u}) = u_{n+1} - F(u_n)$$

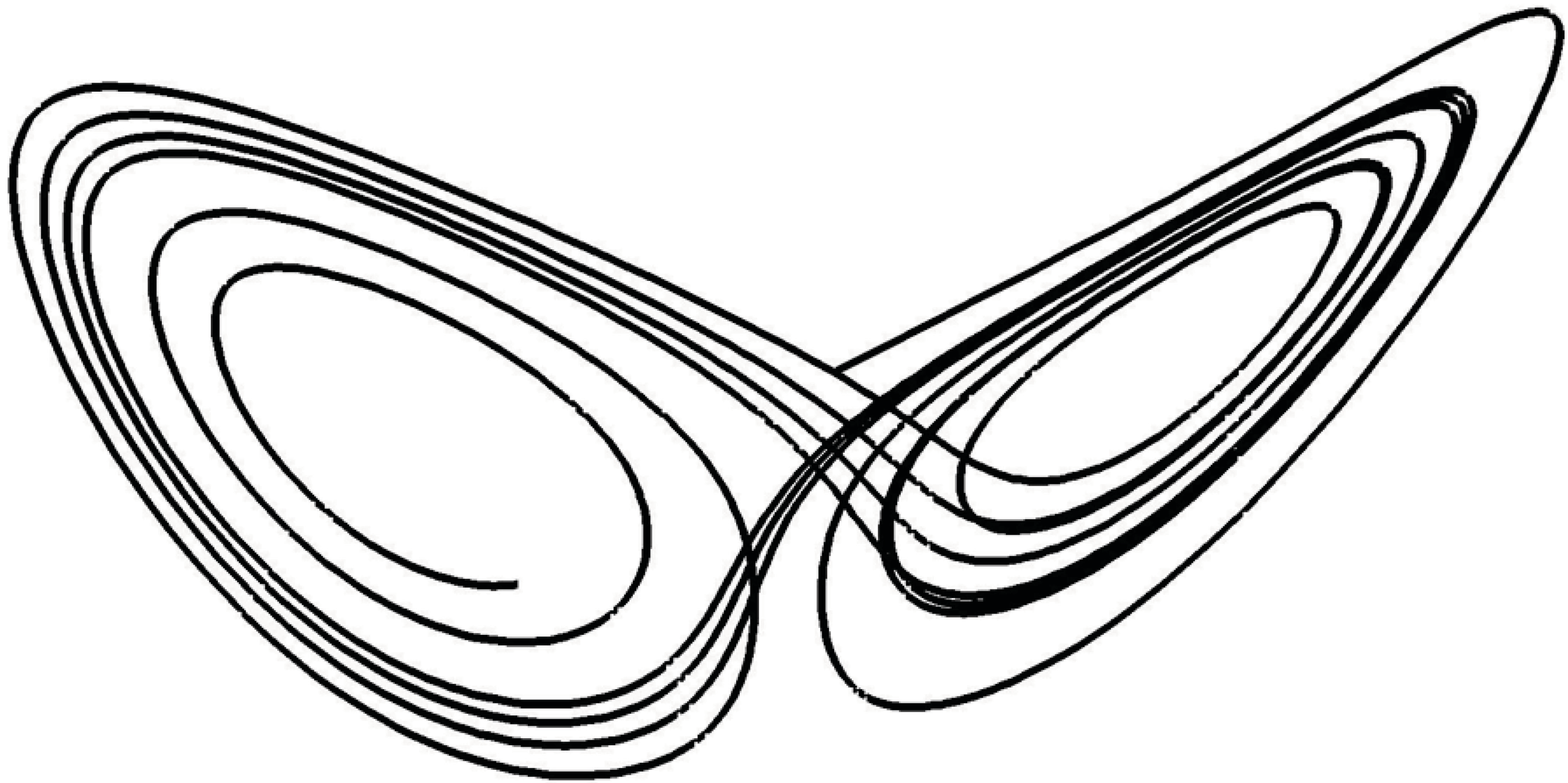
By minimizing:

$$J(u_0) = \|y_n - H(F^n(u_0))\|^2 + \|u_0 - u_{\text{background}}\|^2$$

# Variatonal data assimilation



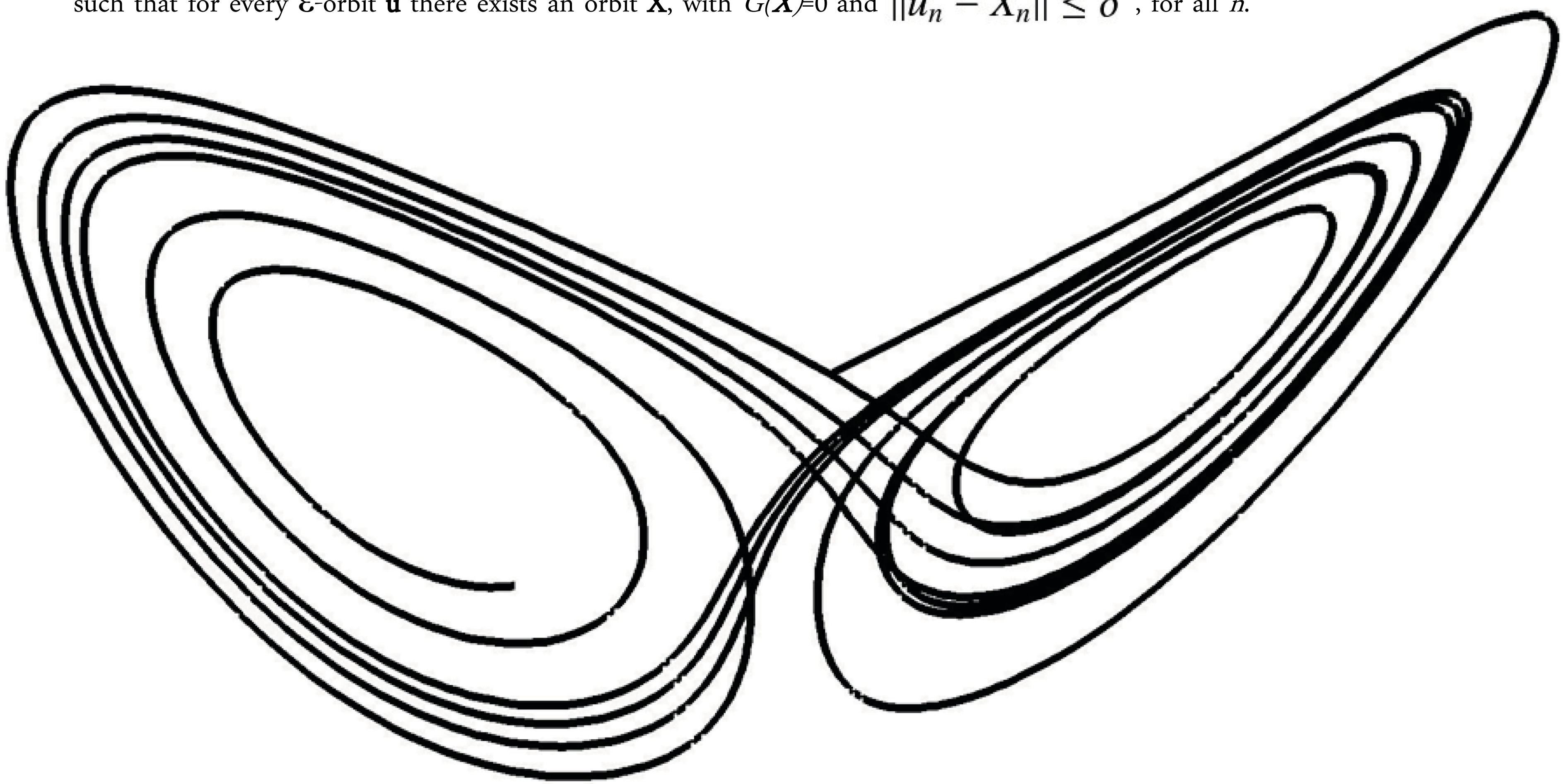
# Shadowing



# Shadowing

If $||G_n(\mathbf{u})|| \leq \epsilon$ , for all *n*, then **u** is called an ε-orbit.

**Shadowing lemma**: in a neighbourhood of a hyperbolic set of the map *F*, for every δ>0 there exists an ε>0, such that for every ε-orbit **u** there exists an orbit **X**, with *G(X)*=0 and $||u_n - X_n|| \leq \delta$ , for all *n*.



# Shadowing

- "Numerical integration of chaotic dynamical systems makes sense"

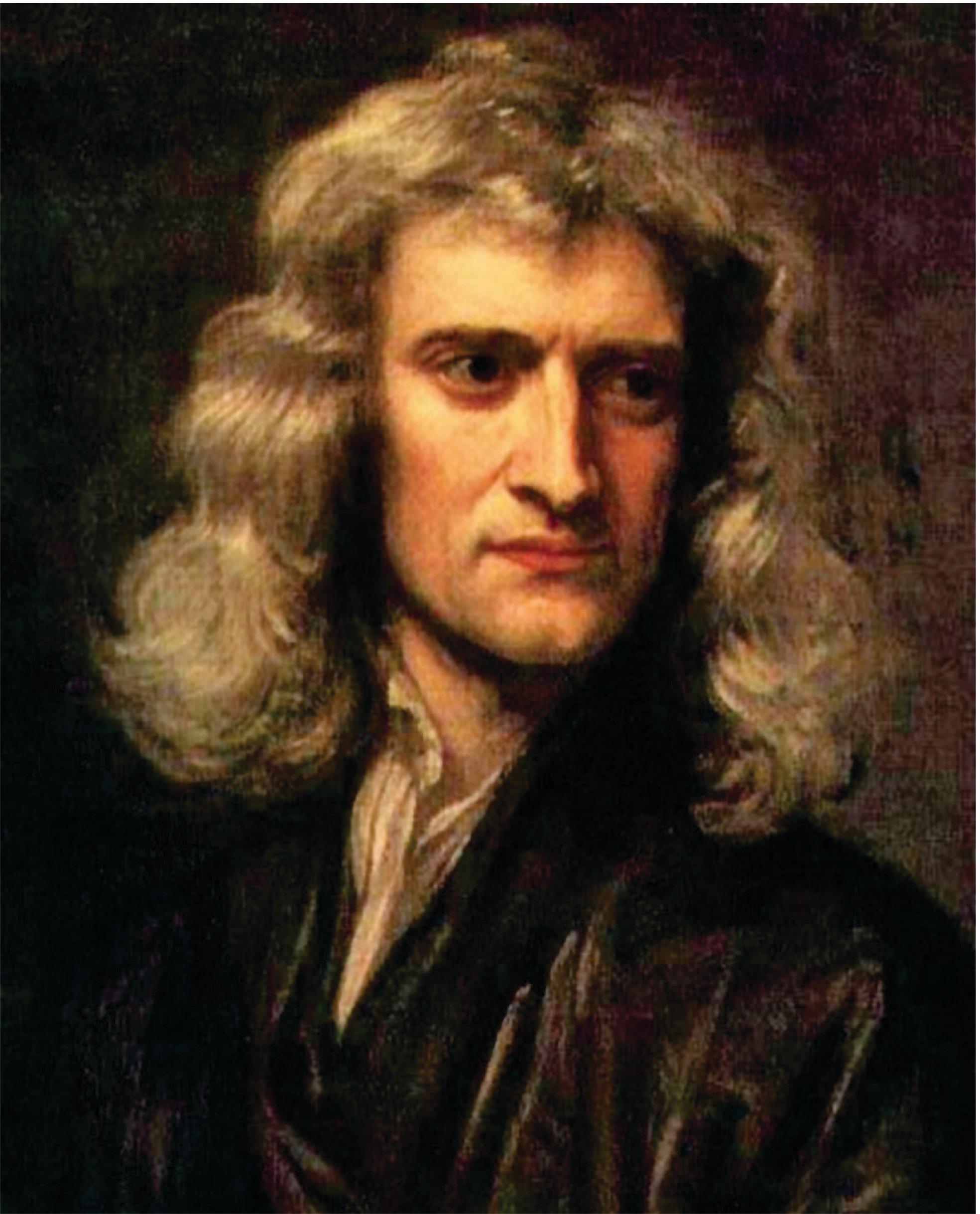
- Note: there is no distinguished point in time! The initial condition $X_0 \neq u_0$

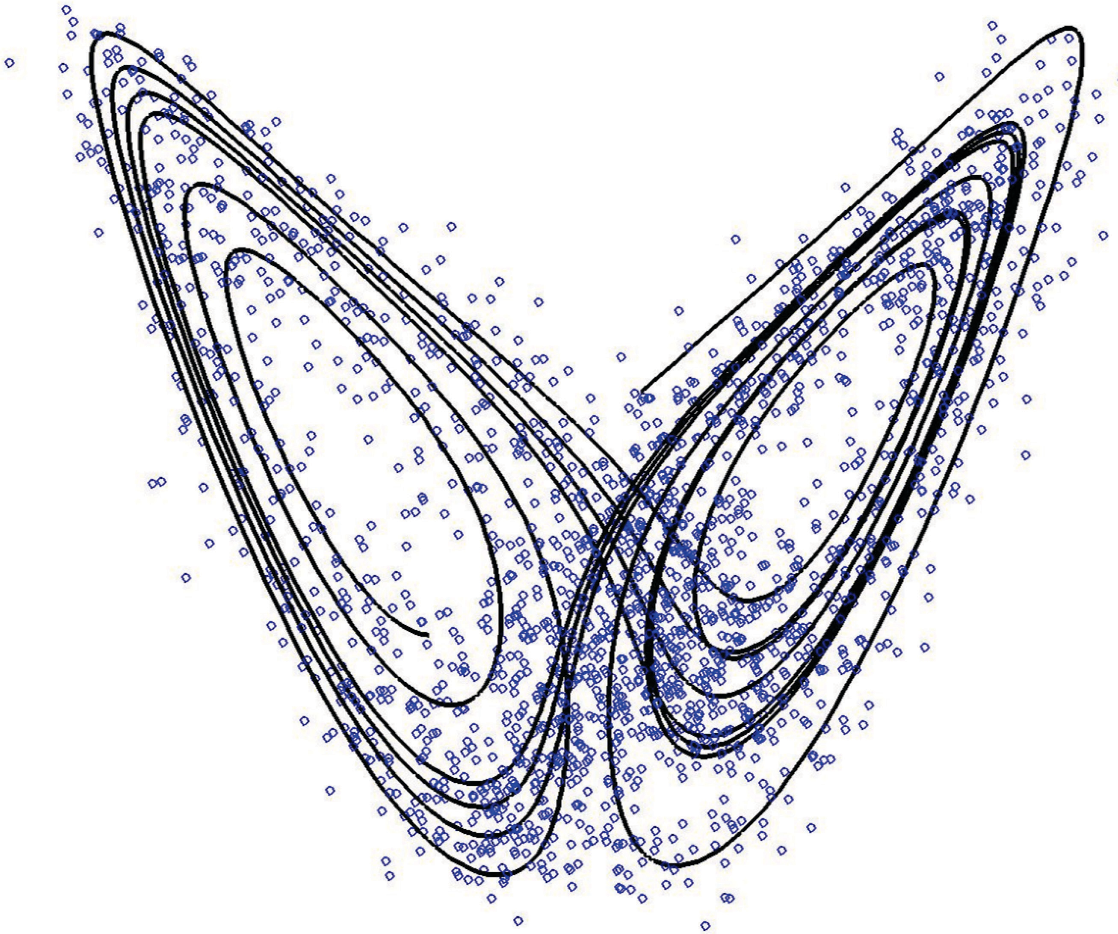
- Shadowing refinement: finding improvements to numerical solutions of a chaotic system

# Shadowing data assimilaton for perfect models

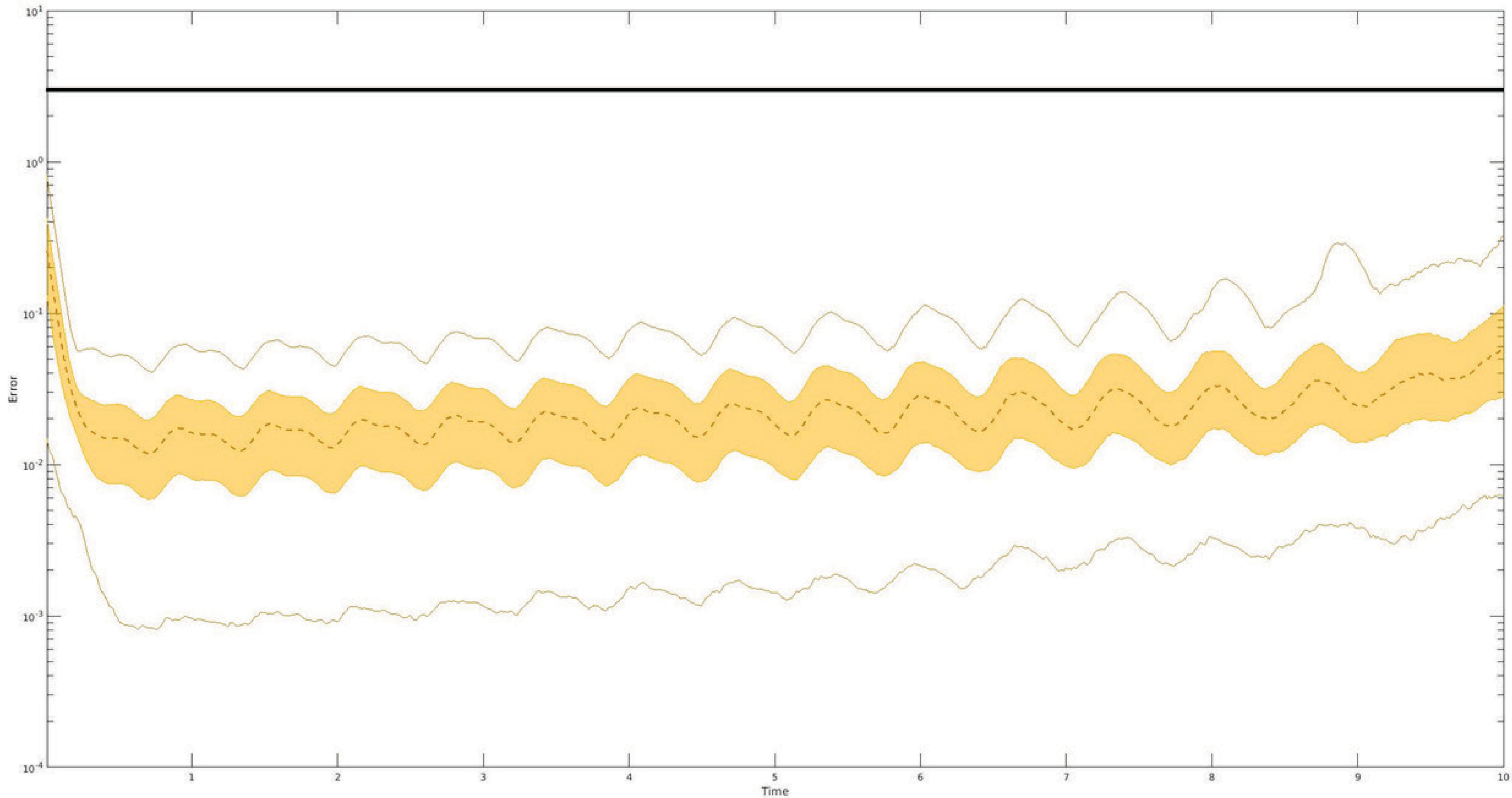
- Idea: start from the observations and use a root finding method to find a solution to $||G(\mathbf{u})|| = 0$ in the vicinity.

# Shadowing data assimilaton
# for perfect models





# Shadowing data assimilaton for perfect models



# Imperfect models: weak 4DVar

- What about model error?
- Example: truth is stochastic differential equation, model is deterministic
- Variational approach: add a model error term to the cost function

$$J = J_{\text{obs}} + J_{\text{background}} + J_{\text{model}}$$

# Imperfect models: weak 4DVar

- Choice of variables:

Model error formulation:

$$J_{\text{model}}(\mathbf{e}) = ||\mathbf{e}||^2, \qquad \text{with } G_n(\mathbf{u}) = e_n$$

State formulation:

$$J_{\text{model}}(\mathbf{u}) = ||G(\mathbf{u})||^2$$

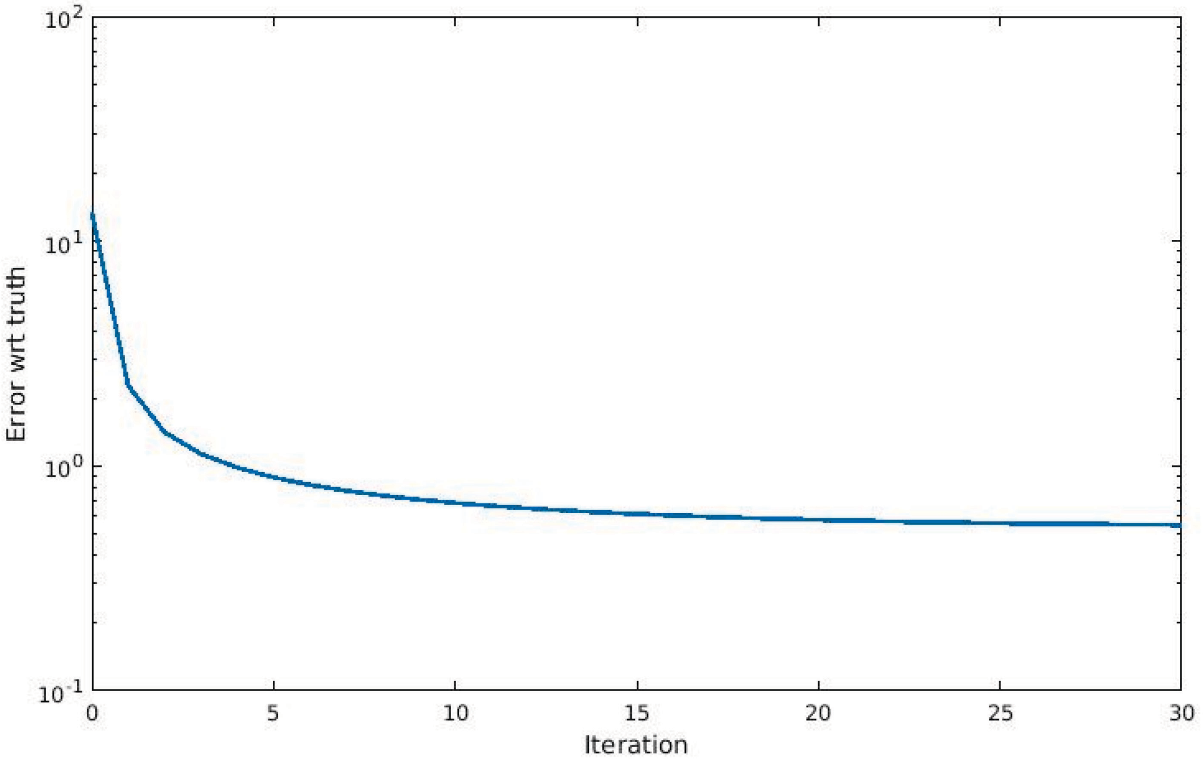
Underdetermined problem: at each step, we have to estimate both the model and observational errors!

# Imperfect models: shadowing

- Existence of a model trajectory close to observations no longer guaranteed.
- Even the closest model trajectory could be incompatible with observations
- Applying perfect model shadowing in the imperfect model scenario may still yield a solution, but far from observations.

# Imperfect models: shadowing

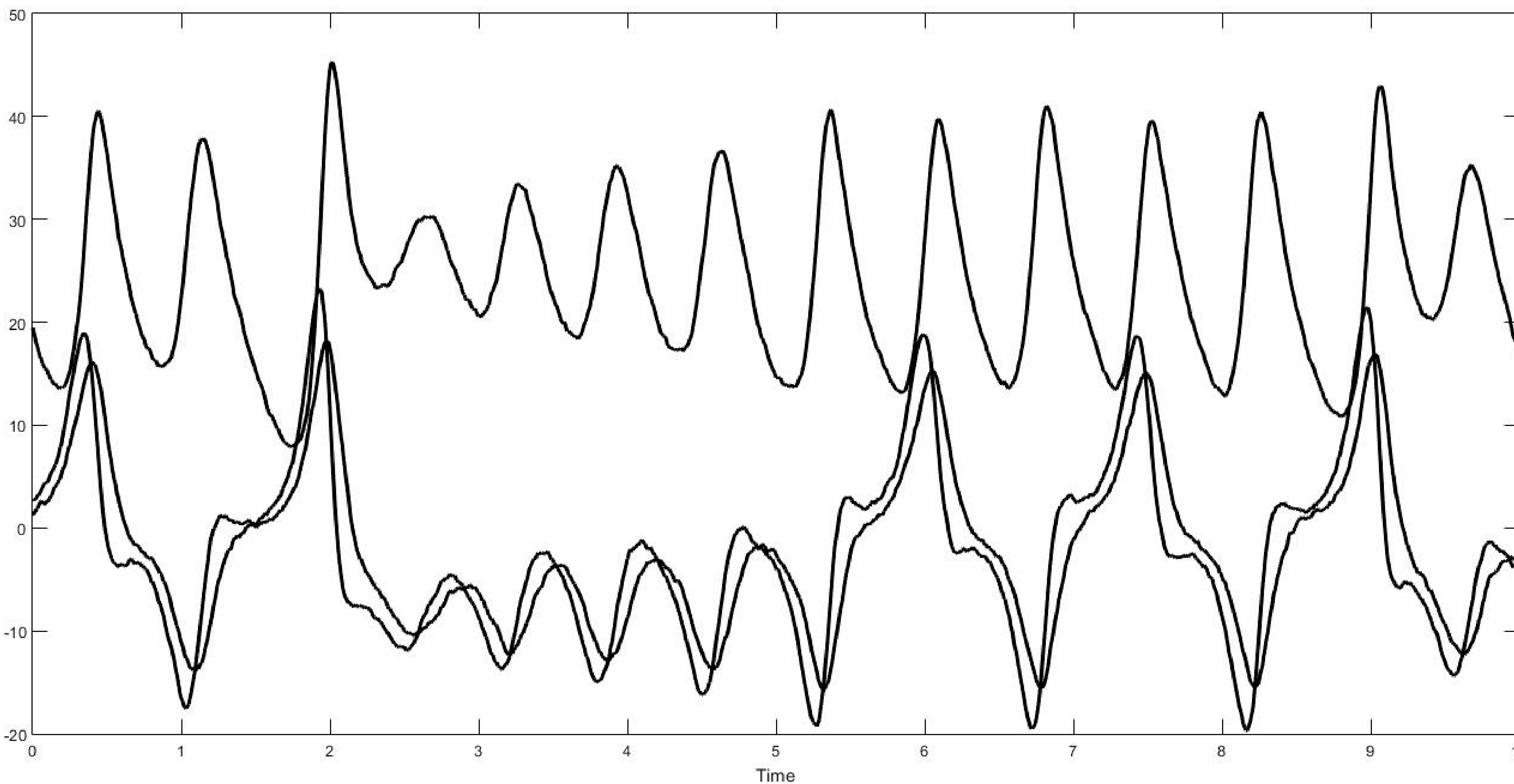
- Idea: regularization (Levenberg-Marquardt)
- Comparable cost and performance to state space weak constraint 4DVar



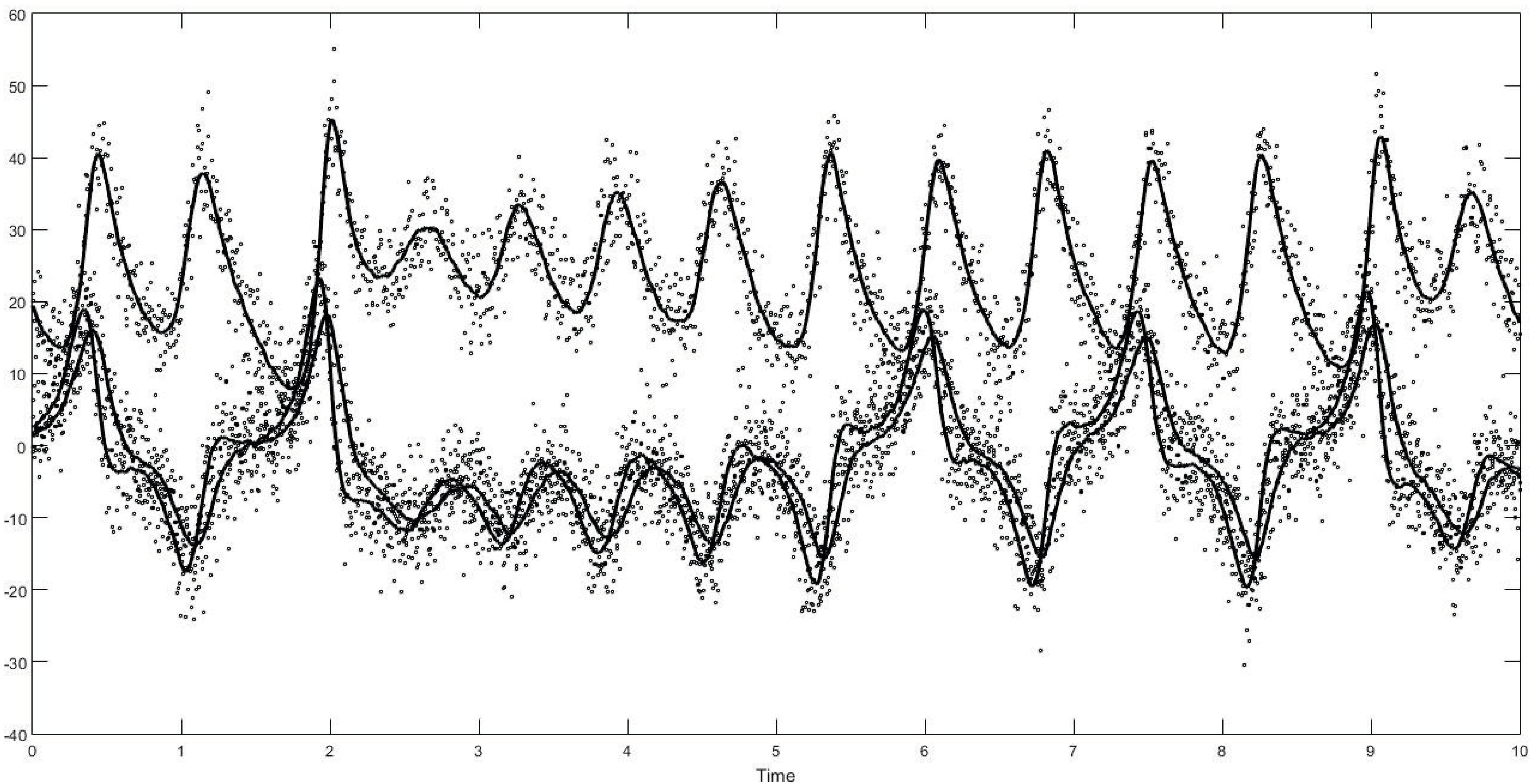
# Example: stochastic L63

- Truth: L63 + Brownian motion forcing
- Imperfect model: L63 without stochastic term
- Observational error sd 2, model error sd 1

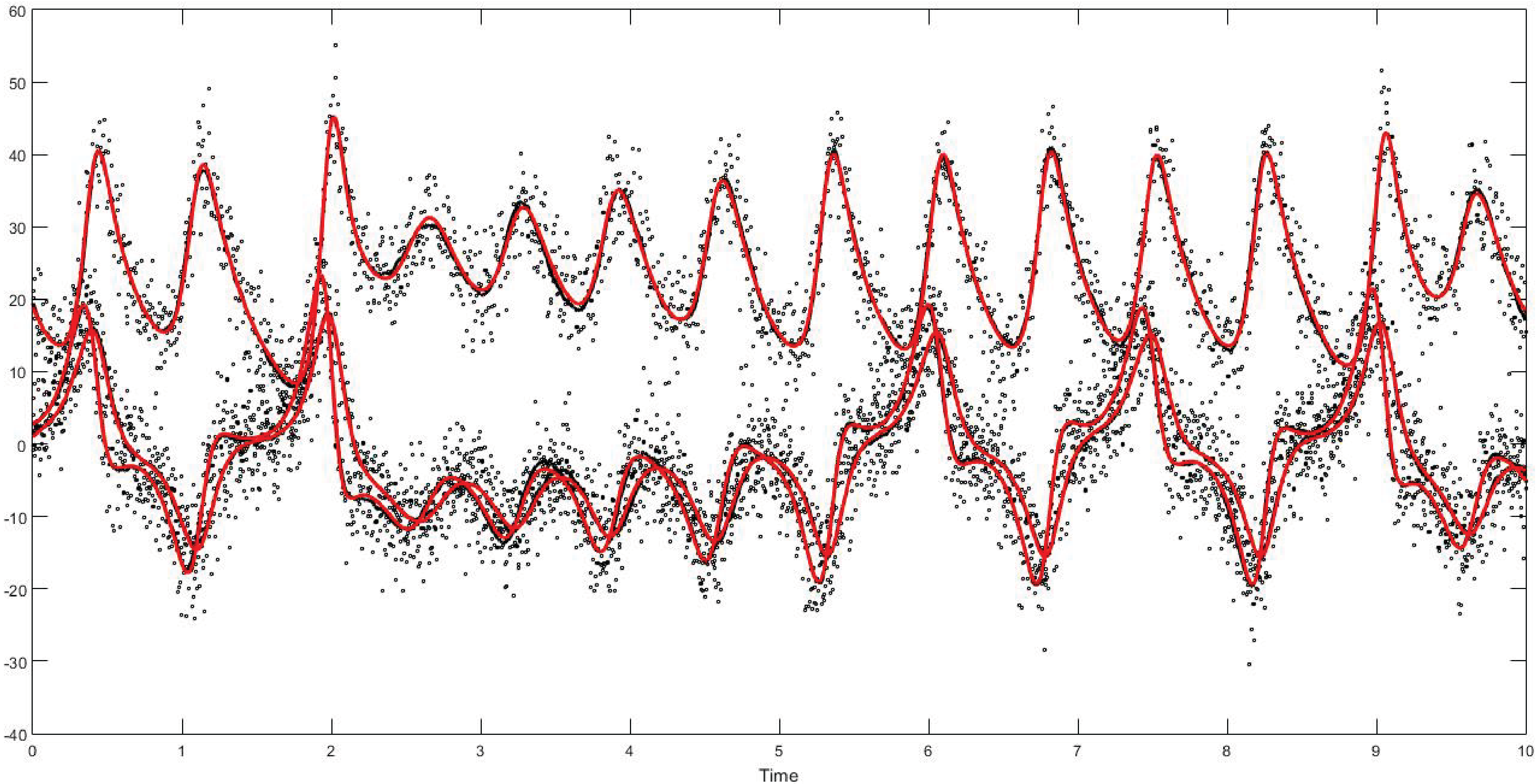
# Stochastic L63: truth



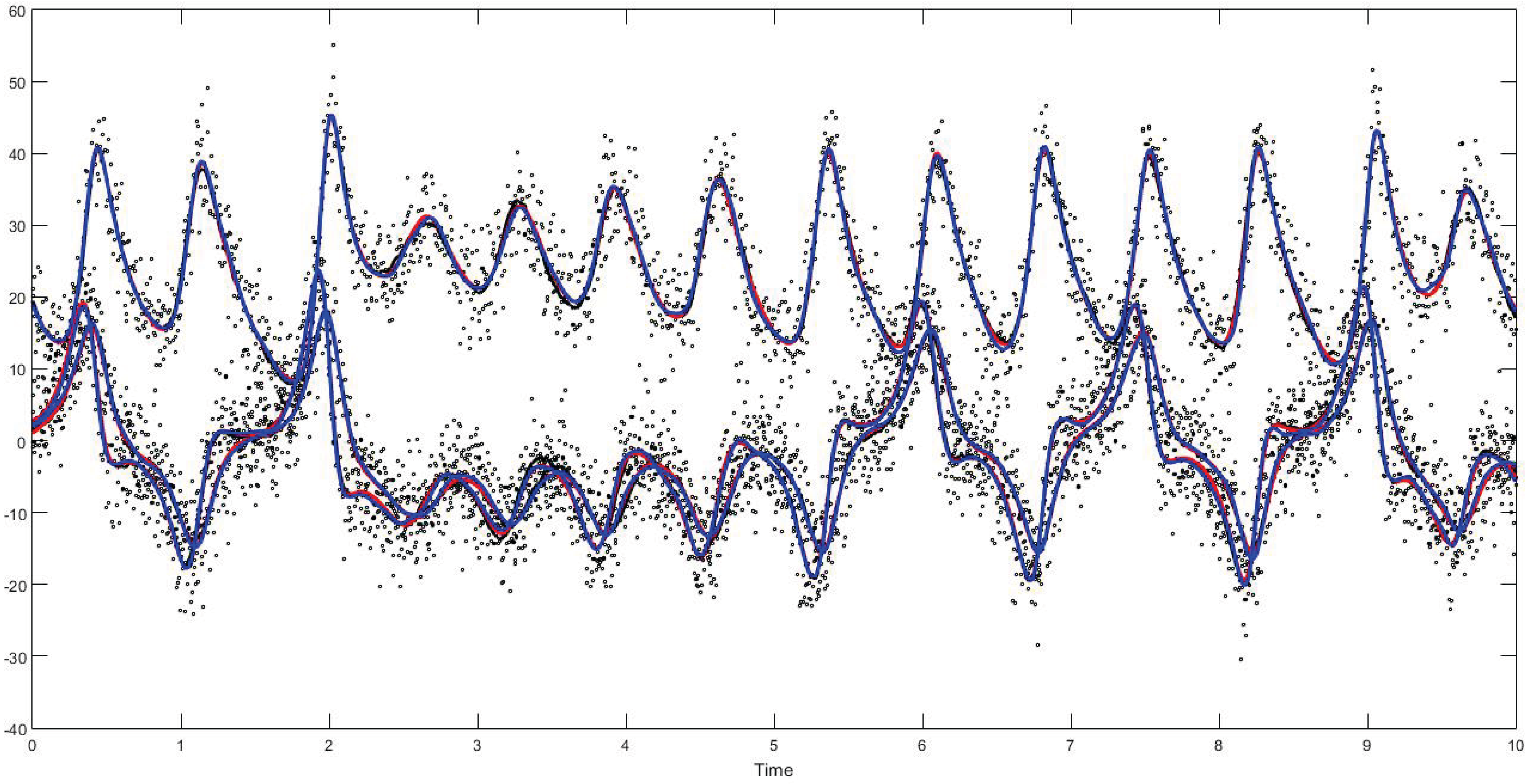
# Stochastic L63: observations



# Stochastic L63: w4DVAR

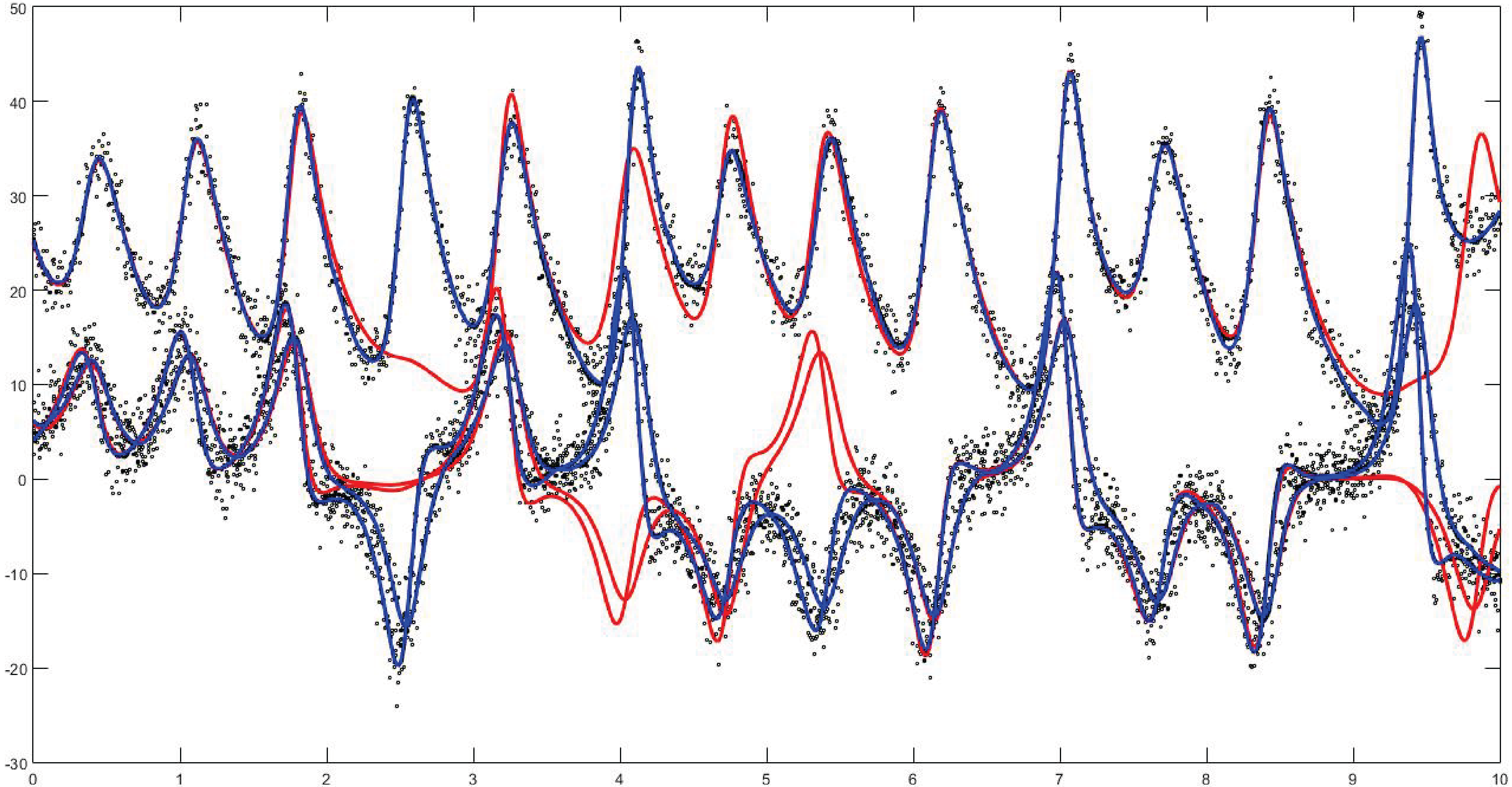


# Stochastic L63: shadowing

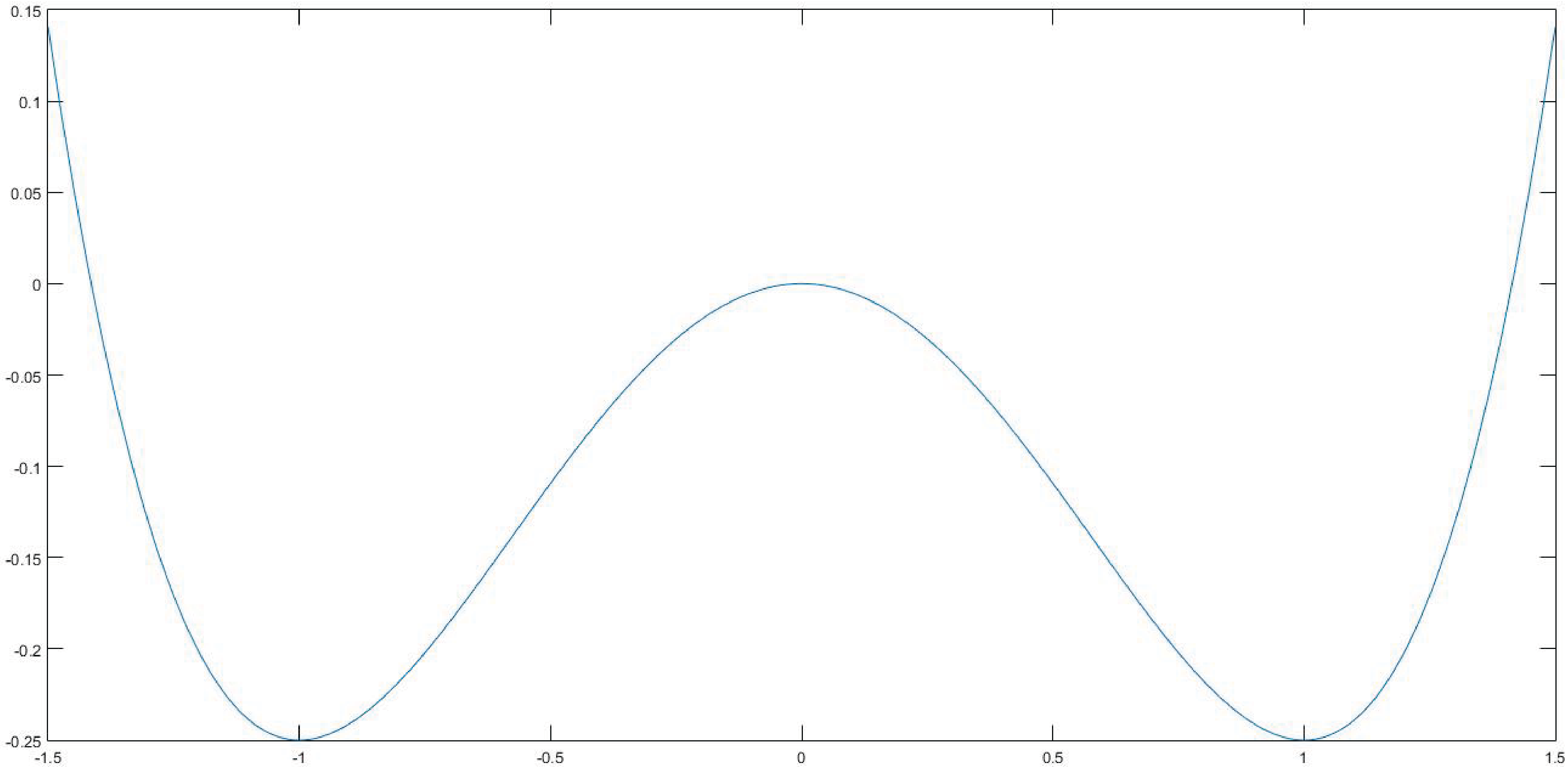


# Stochastic L63: overconfidence in the model

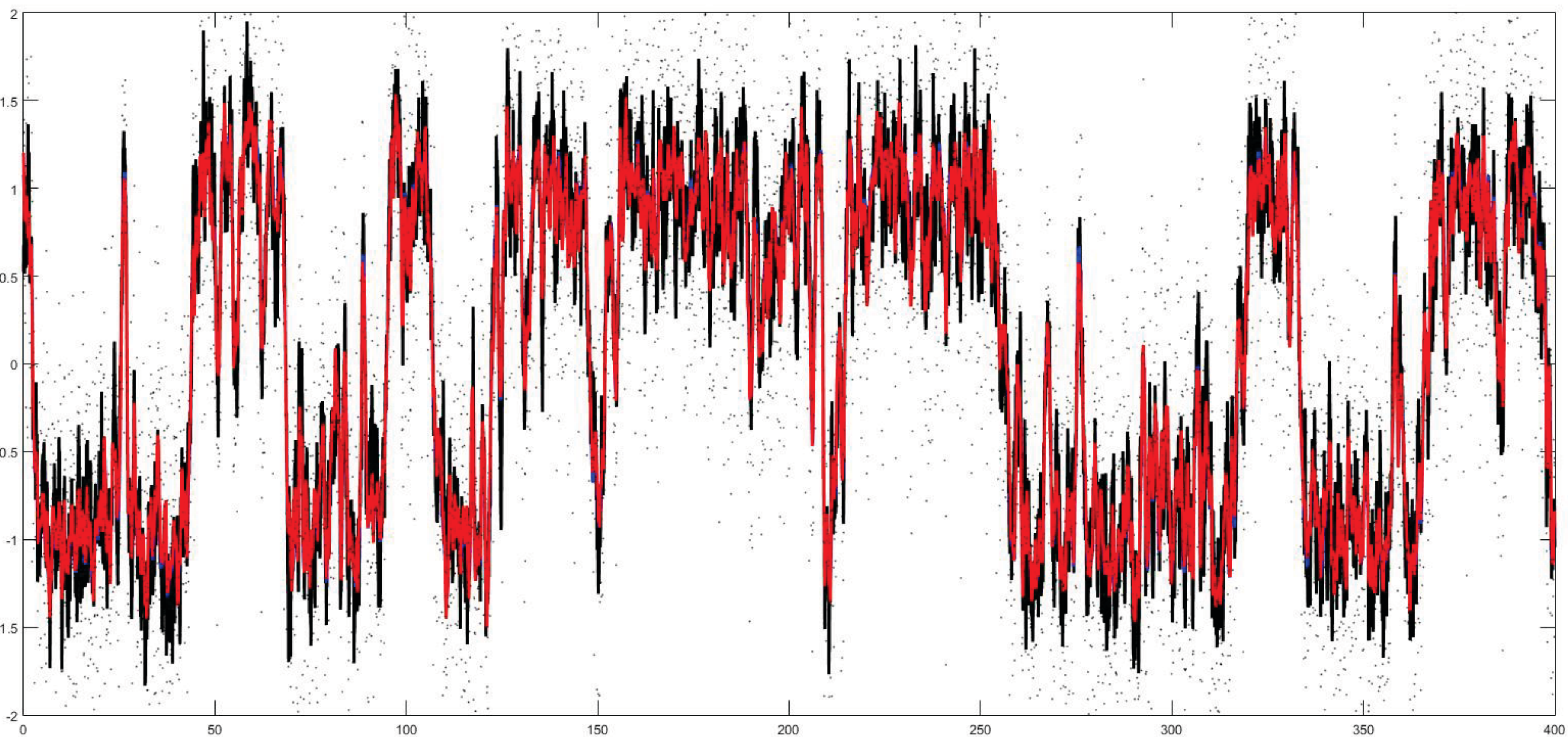
- Model error standard deviation underestimated by a factor 10



# Stochastic double well



# Stochastic double well



# Conclusions

- Shadowing is a powerful technique from dynamical systems theory
- Shadowing may be used to solve variational data assimilation problems
- Shadowing is particularly suitable for highly chaotic systems
- Shadowing can be generalized to deal with imperfect models
- State space weak constraint 4DVar is closely related to shadowing
- Regularizing numerical methods may have benefits over regularizing cost functions

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